Indian Statistical Institute, Bangalore Centre
J.R.F. (I Year) : 2016-2017
Semester I : Supplementary Examination
Analysis - I

26.12.2016 Time: 3 hours. Maximum Marks : 60

Note: Notation and terminology are understood to be as used in class. State clearly the results you are using in your answers.

1. ($3 \times 5 = 15$ marks) Prove or disprove:

   (i) Let $\Omega = \{1, 2, 3, \cdots\}$, $\mathcal{B} =$ power set of Ω, and $\mu(E) =| E |$. Then convergence in $\mu-$measure is equivalent to uniform convergence.

   (ii) Let $(\Omega, \mathcal{B}, \mu)$ be a $\sigma-$finite measure space. Suppose $f_n \to f$ in $L^p(\mu)$, and $g_n \to g$ in $L^q(\mu)$; here $(1/p) + (1/q) = 1$. Then $f_n g_n \to fg$ in $L^1(\mu)$.

   (iii) Let μ denote the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If $f \in L^\infty(\mu))$, then $\| f_x - f \|_\infty \to 0$ as $x \to 0$; here f_x is defined by $f_x(y) = f(x + y), y \in \mathbb{R}$.

2. (10 marks) Let $0 < a, b < \infty$. Define

   $$f(x) \;=\; x^{a-1}(1-x)^{b-1},\; 0 < x < 1.$$

   Show that $f(\cdot)$ is Lebesgue integrable on $(0, 1)$.

3. (15 marks) Let λ be a signed measure on a measurable space $\Omega, \mathcal{B})$. Let $\lambda = \lambda^+ - \lambda^-$ be its Jordan decomposition. For any $E \in \mathcal{B}$, show that

   $$\begin{aligned} \lambda^+(E) &= \sup\{\lambda(F) : F \subset E, F \in \mathcal{B}\}, \\ \lambda^-(E) &= -\inf\{\lambda(F) : F \subset E, F \in \mathcal{B}\}. \end{aligned}$$

4. ($3 + 7 = 10$ marks) Let $1 < p < \infty$, and $q = p/(p-1)$. Let $g \in L^q(\Omega, \mathcal{B}, \mu)$, where μ is a $\sigma-$finite measure. Define $G : L^p(\Omega, \mathcal{B}, \mu) \to \mathbb{R}$ by

   $$G(f) = \int_\Omega f(\omega) g(\omega) d\mu(\omega), \quad f \in L^p(\mu).$$

   (i) Show that G is a bounded linear functional on $L^p(\Omega, \mathcal{B}, \mu)$.

   (ii) Find $\| G \|$.

5. (4 + 6 = 10 marks) Let f, g be nonnegative Borel measurable functions on $\mathbb{R}$; suppose f, g are also integrable with respect to the Lebesgue measure.

(i) Show that the function $(x, y) \mapsto f(x - y)g(y)$ is Borel measurable on $\mathbb{R}^2$.

(ii) Show that $f * g$ is integrable with respect to the Lebesgue measure and find $\| f * g \|_1$.